Lecture 34

Friday, April 2, 2021 1:25 PM

* Prayer
* Spiritual thought
* Answering questions ....

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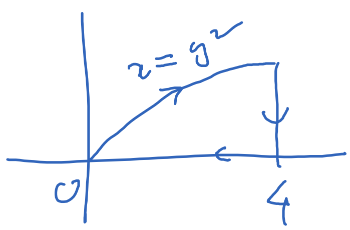
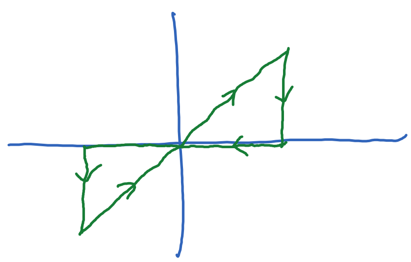
* More examples on Green's theorem

Ex



$$\int_C \left(x^{2/3} + y^2\right) dx$$

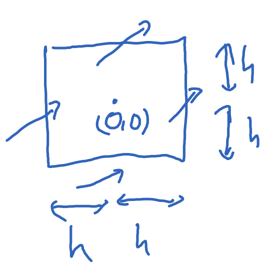
Ex



$$\int_C x\,dx + y\,dy$$

How to explain Green's theorem?

Circulation of a force field / fluid around a curve.

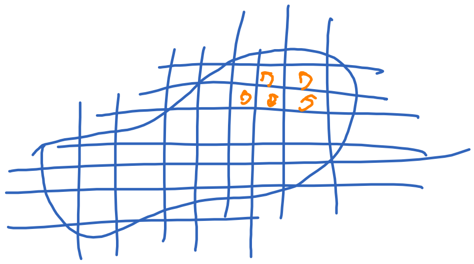


$$\text{Circulation} = \int_C F \cdot dr$$

$$= \int_{-h}^{h} \underbrace{\left(Q(h,t) - Q(-h,t)\right)}_{\approx 2h\,Q_x} dt - \int_{-h}^{h} \underbrace{\left(P(t,h) - P(t,-h)\right)}_{\approx 2h\,P_y} dt$$

$$\approx 4h^2 (Q_x - P_y)$$

$$\text{Circulation density} = \left.\frac{\text{total circulation}}{\text{area}}\right|_{\text{as area} \to 0} = Q_x - P_y$$



$$\int_C F \cdot dr = \text{total circulation}$$

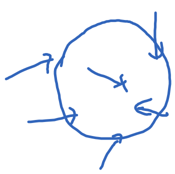
$$= \iint_D (Q_x - P_y)\, dA .$$

Curl and divergence

$f \rightsquigarrow \underbrace{\nabla f}$

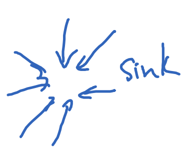
vector field that show the direction f increases the most.

$F \rightsquigarrow \begin{cases} \text{divergence} : \text{how divergent the vector field is at a point} \\ \text{curl} : \text{the direction around which } F \text{ rotates the most.} \end{cases}$



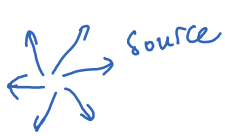
If $\operatorname{div} F > 0$; overall there are more vectors that go out than go in.

If $\operatorname{div} F < 0$; opposite



water is generally incompressible.

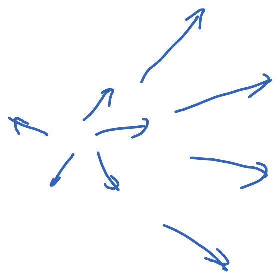
$$\operatorname{div} u = 0$$



$$\operatorname{div} F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \underbrace{\underbrace{\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle}_{\nabla} \cdot \underbrace{\langle P, Q, R\rangle}_{F}}_{\text{"}\nabla \cdot F\text{"}}$$

$F = \langle P, Q, R \rangle$

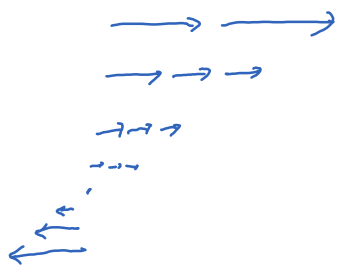
Ex



$F(x,y,z) = \langle x, y, z\rangle$

$\nabla \cdot F = 3$ (const)

Ex



$F(x,y) = \langle y, 0 \rangle$

$\nabla \cdot F = 0$

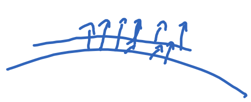
$\downarrow$

Shear flow

Curl $F = \nabla \times F$

$$\begin{array}{cccccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q & R & P & Q \end{array}$$

curl$(\nabla f) = 0$



macroscopic rotation $\neq$ microscopic rotation

$F(x,y) = \langle -y, x, 0\rangle$ has curl $= 2$.

$F(x,y) = \langle y, 0, 0 \rangle$ doesn't seem to rotate in macroscopic scale, but not microscopic scale.

$F(x,y) = \left\langle \dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+y^2}, 0 \right\rangle$ doesn't seem to rotate in microscopic scale, but not in macroscopic scale.